PORTFOLIO OPTIMIZATION AND RISK MEASUREMENT
BASED ON NON-DOMINATED SORTING GENETIC
ALGORITHM

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ABSTRACT. This paper introduces a multi-objective genetic algorithm (MOGA) in regard to the portfolio optimization issue in different risk measures, such as mean-variance, semi-variance, mean-variance-skewness, mean-absolute-deviation and lower-partial-moment to optimize risk of portfolio. This study introduces a PONGSA model by applying the non-dominated sorting genetic algorithm (NSGA-II) to maximize both the expected return and the skewness of portfolio as well as to simultaneously minimize different portfolio risks. The experimental results demonstrated that the PONGSA approach is significantly superior to the GA in all performances, examined such as the coefficient of variation, Sharpe index, Sortino index and portfolio performance index. The statistical significance tests also showed that the NSGA-II models outperformed the GA models in different risk measures.

1. Introduction. Investment portfolio optimization is the process of optimizing the capital proportion of assets held to fit various constraints; it gives the highest return with the least risk. It plays crucial roles in the theory of portfolio selection that provides financial decision-makers and investors with the ability to efficiently manage investment and monitor risks.

The mean-variance (MV) model originally proposed by Markowitz (1952) is a well-known benchmark for portfolio optimization [20]. The traditional MV equilibrium framework models a return on assets as a normal distribution and defines risk as the standard deviation. Both the mean and the variance of returns are only considered in the original capital asset pricing model (CAPM). Furthermore, the upside and downside risks are considered to be equal risk aversion [19].

The MV model may not be the best choice available to investors in terms of an appropriate risk measure. To improve the limitation of MV model, the alternative risk measures such as semi-variance (SV)[21], mean-variance with skewness (MVS) [23], mean absolute deviation (MAD) [16], lower partial moment (LPM) [11][4], etc, have been proposed. Cui et al. [8] Applied MV, MAD, minimax and conditional value at risk (VaR) models based on minimal transaction and cardinality constraint to

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optimize portfolio. The MV model is a bi-criteria optimization equation, whereby a rational investor’s portfolio selection is based on the trade-off between maximum return and minimum risk. Even though the MV model can find an efficient frontier via stochastic programming [27][2], quadratic programming [3][7] and goal programming [6][17][26], the optimization of asset allocation in a portfolio is still a complex and time-consuming NP-hard problem.

Lin and Ko applied GA to optimize the fitness of a portfolio’s asset allocation and VaR. They found that the GA-based portfolio VaR forecasting mechanism could efficiently select more suitable thresholds under the extreme value theory model for each asset in the portfolio [18]. They also used an artificial neural network (ANN) nonlinear, i.e., heuristic search methodologies used to solve portfolio optimization problems in large problem spaces [14]. Chang et al. introduced a GA-based portfolio optimization model with different risk measures, which performed better than an MV model with a small portfolio size [8]. Wilding [28] and Xia, Wang etc. [20] used GA to efficiently construct a portfolio that satisfied multiple objectives. Oh et al. also used GA to measure risks in the portfolio beta instead of the standard deviation in portfolio selection processes [22]. All of the above discussions either used traditional GA to combine multi-objective objectives into a single composite function, or dealt with all objectives individually as the constraint set, because portfolio selection is a multi-objective optimization problem. In practice, it would be difficult to properly select the weight of each objective to characterize the investor’s preferences. However, GA could return a single solution rather than a set of solutions. Investors would prefer a set of fitness solutions (frontier efficient) than a single solution when considering a multi-objective portfolio.

Multi-objective optimization problems must satisfy all of the objectives at a reasonable level without being dominated by any other solution. MOGA applies the nonlinear search capability of GA to efficiently achieve an optimal set. Hussein etc. proposed a MOGA-based risk assessment conceptual framework applied to financial derivative hedging, air traffic conflict detection, that uses scenario simulation to analyze the effect of uncertainty on the multiple conflicting objectives [1]. A famous MOGA methodology, the non-dominated sorting genetic algorithm (NSGA-II) [9], [15], [10], specifies the crowding distance operator to select the non-dominated solutions located in a less crowded region. NSGA-II improves the diversity of the population for a better convergence closer to the true Pareto optimal frontier.

Numerous measures of portfolio methods have been proposed to evaluate the ability of managed fund portfolios to outperform benchmarks. Portfolio performance measurement is the process of comparing the return gained on a portfolio during the investment period. The portfolio performance depends on the superior investment analysts’ capabilities for asset selection and capital weight allocation. The popular portfolio performance indexes include: Sharpe ratio, Treynor index, Jensen index, Sortino ratio [24], portfolio performance index (PPI) [25] etc. These measures have been highly praised by fund management professionals [23][12].

This study presented a PONSGA model that applies NSGA-II methodology to maximize the expected return while simultaneously minimizing the portfolio risk. The portfolio risk measures of five portfolio optimization models specifically proposed were: the MV, SV, VS, MAD and LPM. The experimental results demonstrated that the PONSGA approach was significantly superior to using GA models in all examined portfolio performance indexes, such as the Sharpe index, Sortino index and PPI.
The remainder of this paper is organized as follows. Section 2 describes the PONSGA model and illustrates the various portfolio risk measures. Section 3 introduces the stock selection process analyzes the characteristics of the sample data. Section 4 shows and discusses the experimental results. Finally, Section 5 presents the conclusions.

2. PONSGA model. Portfolio optimization (PO) is a process of optimizing the capital weight of assets held to fit various constraints, such as the highest return, the lowest risk, etc. NSGA is the well-known nonlinear optimization method. NSGA instead of traditional PO method such as quadratic programming is applied to improve the portfolio asset allocation optimization because of its nonlinear search optimization capability. In this study, a PONSGA model was introduced to optimize the asset allocation in a portfolio when considering the maximum return and the minimum risk under different risk measures. The operating procedures of PONSGA, as shown in Figure 1., are described step by step as follows.

Step 1 Prepare the $t$-th generation population ($TP_t$), which is composed of the parent population ($P_t$) and the offspring population ($O_t$). It means $TP_t = P_t \cup O_t$. Each population contains $N$ possible solutions $s_i$, $i=1, 2, \ldots, N$, and the sizes of $P_t$, $O_t$ and $TP_t$ are $N$, $N$ and $2N$, respectively. Initially, the first generation parent population ($P_0$) of size $N$ is randomly created. The standard tournament selection, crossover and mutation operations are applied to generate the offspring population ($O_0$), until size $N$ of $O_0$ is realized.

Step 2 Check whether the terminated criteria, such as the maximum generation and the variation of fitness value are satisfied. If the terminated criteria are achieved, $P_t$ is returned.
**Step 3** Objective models. In the present economy, investors prefer lower risks and higher returns. Five objective models $O_M$ derived from $M = \{MV, SV, MVV, MAD, and LPM\}$ are designed in the PONSGA model, respectively. Each objective model is a combinatorial optimization topic of finding optimal value from a set of objective functions $\mathcal{F}_M = \{\max_W R_M(W), \min_W V_M(W), \max_W S_M(W)\}$, where $R_M(W)$, $V_M(W)$ and $S_M(W)$ are mean, variance and skewness of portfolio return in model $M$. $W$ represents the vector of asset allocation weight, as shown in Eq. (1). The detailed formulations of $R_M(W)$, $V_M(W)$ and $S_M(W)$ are described in Section 2.2.

$$W = [w_1, w_2, ..., w_n]^T$$ (1)

The multi-objective equation can be written as Eq. (2):

$$\mathcal{F}_M = \left\{ \max_W R_M(W), \min_W V_M(W), \max_W S_M(W) \right\}$$ (2)

s.t.

$$\sum_{i=1}^{n} w_i = 1$$

$$w_i \geq 0, i = 1, 2, ..., n$$

**Step 4** All of the solutions $s_i^*, i = 1, 2, ..., 2N$, in the population $TP_i$ are sorted according to each objective function value in ascending order in each model, where the objective functions are denoted as $f(\cdot)$, $f = R, S, V$. The value of $f(\cdot)$ for solution $i$ is $z_f^i$, and $z_f^{\max}$ and $z_f^{\min}$ are the maximum and minimum values of the objective function $f(\cdot)$, respectively.

**Step 5** Non-dominated ranking. Each solution $s_i^*$ in $TP_i$ is ranked according to the density of $s_i^*$ denoted as $D(s_i^*)$ by applying the Pareto-ranking approach. $D(s_i^*)$ is the sum of $D_f(s_i^*)$, $f = R, S, V$, as $D(s_i^*) = D_R(s_i^*) + D_S(s_i^*) + D_V(s_i^*)$, where $D_f(s_i^*)$, as shown in Eq. (3), is calculated by the normalized crowding distance of two solutions: $s_{i-1}^*$ and $s_{i+1}^*$ on both sides of $s_i^*$. $D_f(s_1^*)$ and $D_f(s_{2N}^*)$, with the smallest and largest values, are assigned an infinite value:

$$D_f(s_i^*) = \frac{z_f^{i+1} - z_f^{i-1}}{z_f^{\max} - z_f^{\min}}, i = 2, 3, ..., 2N - 1$$

$$D_f(s_1^*) = D_f(s_{2N}^*) = \infty$$ (3)

The Pareto-ranking is based on the Pareto frontier (PF) which is defined as the set of possible solutions that no further Pareto improvements can be made. The Pareto improvement represents that a change to a different solution ($s_i^*$) that makes at least one individual better off without making any other individual worse off. Figure 2 shows an example of Pareto ranking result. The boxed points represent feasible choices ($s_i^*$) according to different ranking. The Pareto-ranking algorithm is expressed as follows.

function Pareto-ranking ($TP_i$)

Let $P = TP_i$

Let $rank = 0$

while $P \neq \emptyset$

$PO[rank] =$ Find PF set from $P$

$P = P - PO$

$rank++$
These are identified as different non-dominated sets nds_i, i=1,2,...,M. The best non-dominated set nds_1 contains the possible best solutions in TP_i. That is, TP_i = nds_1 \cup nds_2 \cup ... \cup nds_M. Then, all of the member solutions of the sets nds_i, i = 1,2,...,m, are chosen to generate the next parent population P_{i+1}, until P_{t+1} is realized so that |P_{t+1}| \leq N. Let m be the index of a non-dominated front nds_m, then |nds_1 \cup nds_2 \cup ... \cup nds_m| \leq N.

**Step 6** Reproduction. Use the tournament selection method to eliminate the worst solutions from the nds_t fronts through competition to form the P_{t+1} population for the mating pool. Apply crossover and mutation operators to generate offspring population O_{t+1} in the mating pool.

**Step 7** Loop. Go to Step 1 until the terminated criteria are satisfied.

2.1. **Encoding method.** In order to represent a set of asset allocation weights, an individual in the t-th population is divided into n genotypes w_i, i = 1,...,n. Each genotype is a real value encoded to avoid premature convergence and to obtain a higher quality solution with better computation efficiency and robustness. The summation of w_i, i = 1,...,n, is 1.

\[
w_1 + w_2 + w_3 + ... + w_n = 1, 0 \leq w_i \leq 1, w_i \in \mathbb{R}
\]  

(4)

2.2. **Objective models.** In order to find effective portfolio valuation models under maximum return or skewness and minimum risk, the objective models are designed by various portfolio optimization models, such as MV, SV, MVS, MAD and LPM. The models were mathematically represented below.
2.2.1. Mean-variance model. The MV model portfolio optimization problem [20] can be stated as the quadratic programming formula as below. The \( R_{MV}(W) \) and \( V_{MV}(W) \) are two critical objective functions in the portfolio asset allocation problem. First, \( R_{MV}(W) \) denotes the expected return of the portfolio, as shown in Eq. (5):

\[
R_{MV}(W) = W^T \bar{R} = \sum_{i=1}^{n} w_i \bar{r}_i
\]

where \( \bar{R} = [\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n]^T \); \( [\bar{r}_i]_{i=1,2,\ldots,n} \) is the mean of return \( r_i \). \( V_{MV}(W) \) and denotes the variance-covariance of portfolio, as shown in Eq. (6):

\[
V_{MV}(W) = W^T V W
\]

\[
= \sum_{j=1}^{n} W_j^2 \sigma_j^2 + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} w_i w_j \sigma_{ij} (i \neq j), (j > i)
\]

\( \sigma_j^2 \) is the variance of asset \( j \), and \( \sigma_{ij} \) is the covariance between assets \( i \) and \( j \). The objective model \( Q_{MV} \) is shown as follows:

\[
Q_{MV} = \lambda \times V_{MV}(W) - (1 - \lambda) \times R_{MV}(W)
\]

where \( \lambda \) is weighting parameter \((0 \leq \lambda \leq 1)\), which is the parameter of risk aversion.

2.2.2. Semi-variance model. Markowitz proposed the SV model instead of variance-covariance matrix of the quadratic objective function to improve the assumption of an asymmetric return distribution of the MV model [21]. The SV model also considers two objective functions: \( R_{SV}(W) \) and \( V_{SV}(W) \), where \( R_{SV}(W) \) is the same as \( R_{MV}(W) \) and \( V_{SV}(W) \) is shown below:

\[
V_{SV}(W) = \frac{1}{T} \times \sum_{t=1}^{T} (r_{p,t} - \bar{r}_p)^2
\]

where the \( r_{p,t} \) is the portfolio return over the time period \( t = 1, 2, \ldots, T \); \( \bar{r}_p \) is the mean of \( r_{p,t} \). The following equation shows the objective model \( Q_{SV} \):

\[
Q_{SV} = \lambda \times V_{SV}(W) - (1 - \lambda) \times R_{SV}(W)
\]

2.2.3. Mean-variance with skewness model. Maximizing the skewness of return could efficiently improve the performance of the traditional MV model that has been widely discussed in numerous studies [22][28][23]. The MVs model considers \( R_{MVs}(W), V_{MVs}(W) \), and \( S_{MVs}(W) \), simultaneously. \( R_{MVs}(W) \) and \( V_{MVs}(W) \) are the same as \( R_{MV}(W) \) and \( V_{MV}(W) \). The \( S_{MVs}(W) \) represents the skewness of the expected return shown in Eq. (10):

\[
S_{MVs}(W) = E((W^T (R - \bar{R})))^3
\]

\[
= \sum_{i=1}^{n} w_i^3 S_i^3 + 3 \sum_{i=1}^{n} \left[ \sum_{j=i+1}^{n} w_i w_j S_{ij} + \sum_{j=i+1}^{n} w_i w_j^2 S_{jj} \right]
\]

\[
+ 6 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} w_i w_j w_k S_{ijk}
\]
where $S_i$ is the skewness of the expected return of assets $i$, $S_{ij}$, $S_{ijk}$, and $S_{ijk}$ are the co-skewness of the expected return of assets $i$, $j$, or $k$($i=1,2,...,n; j=i+1, i+2,...,i+n; k=j+1, j+2,...j+n$). $Q_{MVS}$ is shown as follows:

$$Q_{MVS} = \lambda \times V_{MVS}(W) - (1 - \lambda - \theta) \times R_{MVS}(W) - \theta \times S_{MVS}(W)$$

where $\lambda$ and $\theta$ are the weighting parameters ($0 \leq \lambda \leq 1, 0 \leq \theta \leq 1$).

2.2.4. Mean absolute deviation model. The MAD model, proposed by Konno and Yamazaki in 1991 [16] is widely used by practitioners because it can reduce time-consumption to solve a large scale portfolio optimization problem with linear programming rather than with quadratic programming as in the case of MV model. The objective model $Q_{MAD}$ considers $R_{MAD}(W)$ and $V_{MAD}(W)$. $V_{MAD}(W)$ is shown in Eq. (12):

$$V_{MAD}(W) = \frac{1}{T} \times \sum_{t=1}^{T} |r_{pt} - \bar{r}_p|$$

$Q_{MAD}$ is shown below.

$$Q_{MAD} = \lambda \times V_{MAD}(W) - (1 - \lambda) \times R_{MAD}(W)$$

2.2.5. Lower partial moment model. Fishburn [11] and Bawa et al. [4] proposed an n-degree lower partial moment to evaluate risk measures characterized by a utility function of portfolio in contrast to the MV model; the higher the degree of LPM measures, the greater the risk aversion of the investor. The $V_{LPM}(W)$ and the objective model $Q_{PLM}$ are shown in Eqs. (14) - (15).

$$V_{LPM}(W) = \frac{1}{T} \sum_{t=1}^{T} [Max(0, \tau - r_{pt})]$$

$$Q_{PLM} = \lambda \times V_{LPM}(W) - (1 - \lambda) \times R_{LPM}(W)$$

where $V_{LPM}(W)$ is the lower partial moment of degree 2; $\tau$ is the investor-target rate of return; $r_{pt}$ is the portfolio return at t period.

3. Data analysis. The data analysis contains data sampling, data exploration, stock selection, descriptive statistics and normality test to illustrate the characteristics of data source.

- Data sampling
  There are 782 companies currently listed in the Taiwan Stock Exchange Corporation (TSE). Among them, 571 companies have been listed for 10 years or more in the Taiwan Economic Journal (TEJ) data bank. These companies were extracted as our experimental targets. The weekly stock return was used to avoid the 7% daily price upward/downward limit in the TSE. There were 570 observations in the data set of each company from January 2000 to December 2010.

- Data exploration
  Exploration of data is generally a time-consuming process. The data collected is not often presented in a ready state that can be transferred into a database table. Sometimes, the sample data collected contains missing data or an unnumbered data set. Missing data is usually expressed as a blank or as NULL data. Unnumbered data normally is described as NAN or N.A. (not available). Both the NULL data and NAN/N.A data were further eliminated


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