A Dynamic Programming Approach to Real Option Valuation in Incomplete Markets

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Abstract—This paper proposes a dynamic programming approach to evaluate uncertain income streams from an investment opportunity in incomplete markets. It is argued that the idea of certainty equivalent (CE) is applied to value such an investment opportunity as well as real options. We show that two approaches to determine the certainty equivalent, the buying price and the selling price approaches, are exactly equal in exponential utility, implying that CE is a fair value for both the buyer and the seller. Numerical techniques are developed to decompose income streams into hedged positions and smaller residuals for convenient computation. The study finds that the dynamic programming approach provides a major advantage over traditional discounted cash flow approach, in that management is not necessarily to estimate risk-adjusted discount rate, thus reducing the chance of making wrong investment decisions.

Keywords—dynamic programming, income streams, incomplete markets, certainty equivalent, real options

I. INTRODUCTION

It is well understood that in complete markets there exists a positive linear state-price function from which a unique risk-neutral probability measure for pricing a contingent claim may be constructed. It is also known that in incomplete markets the risk-neutral probability measures are not unique to such an extent that the price of a contingent claim falls between the price bounds of the infimum and the supremum, which are determined from a collection of all available equivalent martingale measures. Since risk-neutral valuation techniques cannot determine a unique measure for pricing contingent claims in the relaxation of market completeness, recent research is thus directed at incorporating individual’s risk preference and subjective probability measures into the valuation framework of utility maximization. Therefore, the valuation of income streams in incomplete markets may rely on the calculations of a certainty equivalent (CE) either from the buying price approach or the selling price approach.

This paper, drawn from the research on asset pricing in utility maximization frameworks, aims to develop a dynamic programming model for evaluating income streams generated from an investment opportunity in incomplete markets in a discrete-time, discrete-space fashion. The main theme of the paper is that the recursive structures of utility maximization over terminal wealth are explicitly solved in order to evaluate income streams generated from investment opportunity in incomplete markets. Since the risk of income streams in incomplete markets could not be completely hedged away with existing traded securities, the concept of certainty equivalent, also known as the delta property, is introduced to evaluate an investment opportunity. The delta property states that the investor is indifferent between receiving uncertain cash flows and a certain amount. By applying the delta property to utility maximization, we show that the certainty equivalent derived from the buying price approach is equal to that from the selling price approach when the utility function has a delta property. This equality relationship indicates that the certainty equivalent in incomplete markets is a fair price for income streams should both of the buyer and the seller have the same risk attitude.

The rest of the paper is organized as follows: Section 2 reviews the literature on asset pricing in incomplete markets. Section 3 describes the dynamic programming framework from model specifications to derivation of recursive structures. Section 4 presents the classic investment problem in the context of real options. We then solve the classic investment problem with the proposed model and numerical techniques in comparison to traditional discounted cash flow (DCF) approach. Section 5 provides concluding remarks.

II. LITERATURE REVIEW

Since the seminal work in Black and Scholes (1973), most options models along the line of research implicitly or explicitly assume the existence of complete markets which allow for a perfect hedging to eliminate all the states of uncertainty of contingent claims. Harrison and Kreps (1979) consider this fundamental issue of arbitrage argument and find that in complete markets one may construct a risk-neutral probability measure from a linear state-price function for deriving a unique discounted asset price or a martingale. They also make it clear that if the market is complete, the number of traded assets must be at least as many as the number of states of the world. In an infinite-state world with no arbitrage opportunities allowed, Kreps (1981) shows how the martingale property of a price system can be determined in individual’s expected utility representation. Sharing the same spirit of previous studies, Nau and McCardle (1991) postulate that the martingale

1 El-Karoui and Quenez (1995).
price in conventional option pricing models may be still valid as long as the no-arbitrage principle holds.

Empirical evidence suggests that financial markets in reality are incomplete and thus imposing no-arbitrage pricing for a number of reasons. The first reason leading to market incompleteness is mainly caused by trading frictions such as transaction costs. (Aiyagari and Gertler, 1991; Aiyagari, 1993) Furthermore, model risk involved in risk management typically indicates a failure of market completeness. (Figlewski, 1998) Recently, Hugonnier and Morellec (2007) state incomplete markets, in addition to incompleteness of existing assets, also involve management restrictions in that corporate executives are restrained from certain trading strategies. They demonstrate that risk aversion may induce management to expedite investment, leading to an erosion of option value.

To tackle the difficulty of pricing contingent claims in incomplete markets, El-Karoui and Quenez (1995) describe a superhedging technique that the seller may purchase a replicating portfolio at the supremum price to hedge the contingent claim and lock in a profit. Follmer and Leukert (1999) propose an alternative technique called quantile hedging by constructing a hedging strategy which maximizes the probability of a successful hedge under the objective probability measure, given a constraint on the required cost. Another alternative technique is called good-deal pricing proposed by Cochrane and Saa-Requejo (2000). The basic intuition of good-deal pricing is that investors not only exploit all the arbitrage opportunities but also search for hedging the opportunities of good deals expressed by assets with a high Sharpe ratio. Implementing the good-deal hedging would greatly narrow the price bounds in incomplete markets.

Another research approach to asset pricing in incomplete markets is proposed by Smith and Nau (1995), who integrate utility maximization technique and decision tree analysis to evaluate an investment opportunity with the former used to resolve market uncertainty and the latter private uncertainty. This particular assumption of uncertainty makes the integrated framework become a special case in the pricing of incomplete markets. Smith (1998) takes consumption streams into the optimization framework, yet he still makes the same specific assumption regarding project uncertainty as in Smith and Nau (1995). Staum (2004) develops a method of marginal indifference pricing which provides a unique price based on expected utility in the absence of exact hedging replication, but the methodology may be subject to model misspecification. Recently, Pyo (2008) explores the real option problem in incomplete markets within a hyperbolic absolute risk aversion (HARA) utility function to derive narrower price bounds.

Copeland and Antikarov (2001) suggest the Marketed Asset Disclaimer (MAD) assumption, which states that the present value of the project itself, without flexibility, is the best unbiased estimator of the market value of the project if it should be traded on the market. If the twin assets do not exist in reality, theoretically fictitious securities can also be introduced to complete the model and then impose the restrictions to prohibit investors from holding any positions in the fictitious securities. (Pliska, 1997)

III. METHODOLOGY

A. Model Specifications

Suppose a finite sample space, $\Omega$, exists with $k$ ($k<\infty$) elements, $\Omega=[\omega_1,\omega_2,\ldots,\omega_k]$, where $\omega$ denotes the state of the world. All the investment and consumption behavior takes place at time $t \in \{0,1,\ldots,T\}$. There is a probability measure $P$ on $\Omega$, with $P(\omega)>0$ for all $\omega \in \Omega$ and $\sum_{\omega} P(\omega)=1$.

Securities markets are assumed to be frictionless to such an extent that the investor can trade any amount of shares of a security at a market price without incurring any transaction cost. $S$ denotes a security price process, $S=\{S_t:t=0,1,\ldots,T\}$, where $S$ is a security matrix and $S_t$ is a scalar representing the price of security $n$ at time $t$. Among all the securities, $s_t^n$ denotes a risk-free security vector while the others denote risky security vectors. Let $\theta$ be a vector of trading strategies, $\theta=\{\theta_t^n:t=0,1,\ldots,T; n=1,2,\ldots,N\}$, in the investor’s portfolio where the scalar $\theta_t^n$ represents the units of security $n$ held between time $t$ and $t+1$ and the scalar $\theta_t^r$ is the dollar amount invested in $s_t^n$ from time $t$ to $t+1$ at a risk-free rate, $r$. In addition, $x$ denotes a security return process, $x=\{x_t^n:t=0,1,\ldots,T; n=0,1,\ldots,N\}$, where $x_t^n$ is the return of security $n$ from time $t-1$ to time $t$.

Investor’s wealth is denoted by a value process, $V=\{V_t:t=0,\ldots,T\}$, which represents the total value of the portfolio at time $t$. The consumption plans, $C=\{C_t:t=0,\ldots,T\}$, are a non-negative stochastic process with $C_t$ representing the amount of funds consumed by the investor at time $t$. Note that the consumption process is admissible when $C_t \leq V_t$. Also, at the end of the intended time horizon, all the terminal wealth is consumed so that $C_T=0$. Suppose there is a process of cash streams, $H=\{H_t:t=0,\ldots,T\}$ where $H_t(\omega)$ denotes income streams at time $t$ and state $\omega$. Therefore, the wealth, $V_t(\omega)$ ($\forall \omega \in \Omega$, $t \geq 1$), can be expressed as follows:

$$V_t(\omega)=V_{t-1}(\omega)+(1+r)E_{\omega}[S_t(\omega)\cdot H_t(\omega), t \geq 1].$$  \hspace{1cm} (1)

where $r$ is risk-free rate and $x_t^n(\omega)=\frac{S_t(\omega)-S_{t-1}}{S_{t-1}}$.

In the context of investment valuations, an investment opportunity in complete markets is seen as a redundant asset so that its payoffs can be fairly replicated with existing marketed securities. However, in incomplete markets, the payoffs of the investment opportunity cannot be completely hedged away with the marketed securities so that replicating residuals must exist for at least one state of uncertainty. Let $e(\omega)$ denote a vector of replicating residuals in that the scalar $e_i(\omega)$ denote the replicating residual at time $t$ for all $\omega \in \Omega$. Thus, the income streams generated from the investment opportunity can be expressed as follows:
To compare our dynamic programming approach to traditional DCF approach, the value of the deferral option is calculated with the changes in discount rate ranging from 8% to 25%, exhibited in Figure 4. It is obvious that the option value from the DC approach is significantly sensitive to risk-adjusted discount rate, while the option value from our dynamic programming approach is relatively insensitive to risk-adjusted discount rate. The main advantage of the dynamic programming approach over the DCF approach is that management could refrain from making wrongful investment decisions due to over- and under-estimating project risk and thus risk-adjusted discount rate.

Figure 4. The Certainty Equivalent of Three Alternatives

V. CONCLUDING REMARKS

In this paper, a dynamic programming model in a discrete-time fashion is presented to value an investment opportunity by maximizing expected utility over terminal wealth. Since the equivalent martingale price does not exist in incomplete markets, the investment opportunity must be valued by a certainty equivalent. It is then demonstrated that two approaches to deriving certainty equivalent, the buying price approach and the seller price approach, are exactly equal in the exponential utility, given that the buyer and the seller have same risk preference. This equality implies that the certainty equivalent can be a fair price of income streams for both the buyer and the seller. The model also finds that certainty equivalent tends to increase, as the investment decision-maker becomes more risk-averse.

In the utility maximization models, numerical techniques is also developed in order to resolve income streams by dividing them into two components: the hedged positions under the optimal trading strategies and the replicating residuals, the former representing the positions that can be hedged with existing traded securities, and the latter being the residuals that cannot be hedged away. Since the replicating residuals disappear in complete markets, the certainty equivalent of income streams converges to the value of the hedged position. In other words, in complete markets the objective probability measure becomes the risk-neutral probability measure, which leads the certainty equivalent to the martingale price. Hence, the certainty equivalent converges in complete markets to the martingale price and diverges in incomplete markets into an interval of prices which are conditional on the replicating residuals, a representation of the degree of market incompleteness. The utility maximization model thus becomes a general valuation framework which may be applied either in complete or incomplete markets.

In the application of solving the extended investment problem, it is found that the dynamic programming approach could be readily used to evaluate all investment alternatives available to management. It is demonstrated that management’s risk attitude in incomplete markets could influence the option value in the utility maximization, although the impact is contingent on the degree of market incompleteness. The study finds that the dynamic programming approach provides a major advantage over traditional DCF approach without estimating risk-adjusted discount rate, thus avoiding making wrongful investment decisions especially for a near-zero-NPV project.

REFERENCES