Algebraic criterion for robust controllability of continuous linear time-delay systems with parametric uncertainties

Shinn-Horng Chen\textsuperscript{a}, Jyh-Horng Chou\textsuperscript{a,b,c,*}

\textsuperscript{a}Department of Mechanical/Electrical Engineering, National Kaohsiung University of Applied Sciences, 415 Chien-Kung Road, Kaohsiung 807, Taiwan, Republic of China

\textsuperscript{b}Institute of Electrical Engineering as well as Department of Mechanical and Automation Engineering, National Kaohsiung First University of Science and Technology, 1 University Road, Yenchao, Kaohsiung 824, Taiwan, Republic of China

\textsuperscript{c}Department of Healthcare Administration and Medical Informatics, Kaohsiung Medical University, 100 Shi-Chuan 1st Road, Kaohsiung 807, Taiwan, Republic of China

Received 19 September 2011; received in revised form 16 February 2013; accepted 12 June 2013
Available online 22 June 2013

Abstract

The robust controllability problem for the continuous linear time-delay systems with structured parametric uncertainties is studied in this paper. A new sufficient algebraic criterion for the robust controllability of uncertain linear time-delay systems is established. The proposed sufficient condition can provide the explicit relationship of the bounds on system uncertainties for guaranteeing the controllability property. Three numerical examples are given to illustrate the application of the proposed sufficient algebraic criterion and to compare the results with those obtained from the approaches in the literature.

© 2013 The Franklin Institute. Published by Elsevier Ltd. All rights reserved.

1. Introduction

It is well recognized that time-delay phenomenon is ubiquitous in nature and engineering, including mechanical engineering, aeronautics and astronautics, ecology, biology, information
technology, economics, and so on. Time-delay effect may occur naturally because of the inherent characteristics of some system components, or part of the control process [9]. Besides, controllability plays a central role throughout the history of modern control theory and engineering because it has close connection to eigenvalue assignment, optimal control, and controller design [14]. Therefore, the controllability problem of continuous linear time-delay systems has been studied by some researchers (see, for example, [20,21,10,12,13,22,18,9,19]). On the other hand, in fact, in many cases it is very difficult, if not impossible, to obtain the accurate values of some system parameters. This is due to the inaccurate measurement, unaccessibility to the system parameters, or variation of the parameters. These system uncertainties may destroy the controllability property of the linear time-delay systems. By using the result presented by Hewer [10] as well as Levsen and Nazaroff [12], the robust controllability problem of uncertain continuous linear time-delay systems can be transferred into that of uncertain continuous linear delay-free systems. Some methods for studying the robust controllability problem of uncertain continuous linear delay-free systems have been proposed by some researchers (see, for example, [15,8,6,4,17,2]; and therein references). Therefore, these proposed methods can be used to solve the robust controllability problem of uncertain continuous linear time-delay systems. Most notably, the approaches proposed by Elizondo and Collado [8], Cheng and Zhang [4] as well as Chen and Chou [2] give algebraically elegant derivations. However, the approach of Elizondo and Collado [8] is only suitable for the unidirectional uncertainty case, the parametric uncertainties considered by Cheng and Zhang [4] must satisfy the sign-invariant condition, and all the uncertain elements of the interval matrices considered by Chen and Chou [2] must have the same variations.

The purpose of this paper is to present an alternative new approach for investigating the robust controllability problem of the continuous linear time-delay systems with system uncertainties. The presented new approach provides an algebraically elegant derivation. A new sufficient algebraic criterion is proposed to guarantee the robust controllability of uncertain linear time-delay systems. The robust controllability problem for uncertain continuous linear delay-free systems studied in this paper is more general than those considered by Elizondo and Collado [8] as well as Cheng and Zhang [4]. The mathematical means used in this paper is also very different from those used by Elizondo and Collado [8] as well as Cheng and Zhang [4]. The proposed sufficient condition can provide the explicit relationship of the bounds on system uncertainties for having the robust controllability property. Three numerical examples are also given in this paper to illustrate the application of the proposed sufficient algebraic criterion and to compare the results with those obtained from the approaches of Elizondo and Collado [8] and Cheng and Zhang [4].

2. Robust controllability analysis

Consider the following continuous linear time-delay system with system uncertainties:

\[
\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)x(t-\tau) + (C + \Delta C)u(t), \quad t > t_0,
\]

where \( x(t) \in \mathbb{R}^n \) is the system state vector, \( u(t) \in \mathbb{R}^m \) is the input vector, \( \tau > 0 \) denotes the time delay, \( A, B \) and \( C \) are, respectively, the \( n \times n, n \times n \) and \( n \times m \) constant matrices, as well as \( \Delta A, \Delta B \) and \( \Delta C \) are, respectively, the uncertain matrices existing in the system matrices \( A \) and \( B \), and in the input matrix \( C \) due to the inaccurate measurement, unaccessibility to the system parameters, or variation of the parameters. Let \( B \) be the Banach space of real \( n \)-vector-valued continuous functions defined on the interval \([t_0-\tau, t_0]\) with the uniform norm, i.e., if \( \Phi \in B \), we
have \( \| \Phi \| = \max_{t \in [t_0 - \tau, t_0]} |\Phi(t)| \). The initial function space is assumed to be \( B \), the space of continuous functions mapping \([t_0 - \tau, t_0] \) into \( R^n \), and the \( R^m \)-valued control function \( u(t) \) is measurable and bounded on every finite time interval [21].

In many interesting problems, we have only a small number of uncertainties, but these uncertainties may enter into many entries of the system and input matrices [23,1,16]. For example, consider a two-mass system with an uncertain stiffness given by Sinha [16]. The system matrix \( A \) is

\[
A = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-\tilde{k} & \tilde{k} & 0 & 0 \\
\tilde{k} & -\tilde{k} & 0 & 0 \\
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-k_0 & k_0 & 0 & 0 \\
k_0 & -k_0 & 0 & 0 \\
\end{bmatrix} + \epsilon \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
\end{bmatrix},
\]

where the parametric uncertainty \( \epsilon \) enters into four entries of the system matrix. So, the forms are the elemental parametric uncertain forms for the general and practical cases [16]. Therefore, in this paper, we suppose that the uncertain matrices \( \Delta A, \Delta B \) and \( \Delta C \) take the forms

\[
\Delta A = \sum_{k=1}^{m} \epsilon_k A_k, \quad \Delta B = \sum_{k=1}^{m} \epsilon_k B_k, \text{ and } \Delta C = \sum_{k=1}^{m} \epsilon_k C_k,
\]

where \( \epsilon_k (k = 1, 2, \ldots, m) \) are the elemental parametric uncertainties, as well as \( A_k, B_k \) and \( C_k \) are, respectively, the given \( n \times n, n \times n \) and \( n \times m \) constant matrices which are prescribed a prior to denote the linearly dependent information on the elemental parametric uncertainties \( \epsilon_k \), in which \( k = 1, 2, \ldots, m \).

Before we investigate the robust property of controllability for the uncertain linear time-delay system in Eqs. (1) and (2), the following definitions and lemmas need to be introduced first.

**Definition 1. Weiss [21]**

The system \( \dot{x}(t) = Ax(t) + Bx(t-\tau) + Cu(t) \) with \( t > t_0 \) is controllable to the origin from time \( t_0 \) if, for each \( \Phi \in B \), there exists a finite time \( t_1 > t_0 \) and an admissible input \( u(t) \) defined on \([t_0, t_1]\) such that \( x(t_1, t_0, \Phi, u) = 0 \), where \( x(t_1, t_0, \Phi, u) \) denotes a solution to \( \dot{x}(t) = Ax(t) + Bx(t-\tau) + Cu(t) \) at time \( t_1 \) corresponding to initial time \( t_0 \), initial function \( \Phi \in B \) and input \( u(t) \).

**Definition 2. Desoer and Vidyasagar [7]**

The measure of a matrix \( W \in C^{n \times n} \) is defined as

\[
\mu(W) \equiv \lim_{\theta \to 0} \frac{\|I + \theta W\| - 1}{\theta},
\]

where \( \| \cdot \| \) is the induced matrix norm on \( C^{n \times n} \).

**Lemma 1. Hewer [10], and Levens and Nazaroff [12]**

*If the system \( \dot{x}(t) = (A + B)x(t) + Cu(t) \) with \( t > t_0 \) is controllable, then the linear time-delay system \( \dot{x}(t) = Ax(t) + Bx(t-\tau) + Cu(t) \) with \( t > t_0 \) is controllable in sense of Weiss [21] for any \( \tau > 0 \).*
Lemma 2. For any $\tau > 0$, the linear time-delay system $\dot{x}(t) = Ax(t) + Bx(t-\tau) + Cu(t)$ with $t > t_0$ is controllable in sense of Weiss [21], if the following $n^2 \times n(n+m-1)$ controllability matrix

$$Q = \begin{bmatrix}
I_n & 0 & \cdots & 0 & 0 & \cdots & 0 & C \\
-(A+B) & I_n & \cdots & \cdots & 0 & \cdots & 0 & C \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}$$

has rank $n^2$, where $A, B \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{n \times m}$ and $I_n$ denotes the $n \times n$ identity matrix.

Proof. Following the similar proof procedure as that in the work of Chen and Chou [3], in the above matrix $Q$ of Eq. (3), add $(A+B)$ times the first (block) row to the second, then add $(A+B)$ times the second row to the third, and so on. The result is a matrix

$$Q = \begin{bmatrix}
I_n & 0 & \cdots & 0 & 0 & \cdots & 0 & C \\
0 & I_n & \cdots & \cdots & 0 & \cdots & 0 & C \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}$$

The controllability matrix $\begin{bmatrix} C & (A+B)C & \cdots & \cdots & (A+B)^{n-2}C \\
(A+B)C & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}$ is of rank $n$ if and only if the matrix in Eq. (4) has rank $n^2$ (i.e., the matrix in Eq. (3) has rank $n^2$). And, the system $\dot{x}(t) = (A+B)x(t) + Cu(t)$ with $t > t_0$ is controllable, if and only if $\text{rank}(\begin{bmatrix} C & (A+B)C & \cdots & \cdots & (A+B)^{n-1}C \end{bmatrix}) = n$ [11]. So, from Lemma 1, we can conclude that, if the matrix in Eq. (3) has rank $n^2$, then, for any $\tau > 0$, the linear time-delay system $\dot{x}(t) = Ax(t) + Bx(t-\tau) + Cu(t)$ with $t > t_0$ is controllable in sense of Weiss [21]. Q.E.D.


The matrix measures of the matrices $\overline{W}$ and $\overline{V}$, namely $\mu(\overline{W})$ and $\mu(\overline{V})$ respectively, are well defined for any norm and have the following properties:

(i) $\mu(\pm I) = \pm 1$, for the identity matrix $I$;
(ii) $-\|W\| \leq -\mu(-W) \leq \mu(W) \leq \|W\|$, for any norm $\|\cdot\|$ and any matrix $W \in \mathbb{C}^{n \times n}$;
(iii) $\mu(W + V) \leq \mu(W) + \mu(V)$, for any two matrices $W, V \in \mathbb{C}^{n \times n}$;
(iv) $\mu(\gamma W) = \gamma \mu(W)$, for any matrix $W \in \mathbb{C}^{n \times n}$ and any non-negative real number $\gamma$;

where $\lambda(W)$ denotes any eigenvalue of $W$, and $\text{Re}(\lambda(W))$ denotes the real part of $\lambda(W)$.

Lemma 4. For any $\gamma < 0$ and any matrix $W \in \mathbb{C}^{n \times n}$, $\mu(\gamma W) = -\gamma \mu(-W)$.

Proof. This lemma can be immediately obtained from the property (iv) in Lemma 3. Q.E.D.
Lemma 5. Chen and Chou [3]

Let $N \in C^{n \times n}$. If $\mu(-N) < 1$, then $\det(I + N) \neq 0$.

Proof. From the property (ii) in Lemma 3 and since $\mu(-N) < 1$, we can get that $\text{Re}(\lambda(N)) \geq -\mu(-N) > -1$. This implies that $\lambda(\bar{N}) \neq -1$. So, we have the stated result. Q.E.D.

From Lemma 2, it is known that, for any $\tau > 0$, the uncertain linear time-delay system in Eqs. (1) and (2) is robustly controllable in sense of Weiss [21], if the following $n^2 \times n(n + m - 1)$ matrix

$$
\tilde{Q} = Q + \sum_{k=1}^{m} \epsilon_k E_k
$$

(5)

has full row rank $n^2$, where $Q$ is given in Eq. (3), and

$$
E_k = 
\begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & C_k \\
-(A_k + B_k) & 0 & \cdots & 0 & 0 & \cdots & 0 & C_k \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & -(A_k + B_k) & C_k & \cdots & 0 & 0
\end{bmatrix}.
$$

(6)

Let the singular value decomposition of $Q$, which has rank $n^2$, be

$$
Q = U \left[ S \ 0_{n^2 \times n(n-1)} \right] V^H,
$$

(7)

where $U \in \mathbb{R}^{n^2 \times n^2}$ and $V \in \mathbb{R}^{n(n+m-1) \times n(n+m-1)}$ are the unitary matrices, $V^H$ denotes the complex-conjugate transpose of matrix $V$, $S = \text{diag}[\sigma_1, \ldots, \sigma_{n^2}]$, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n^2} > 0$ are the singular values of $Q$.

In what follows, we present a sufficient criterion for ensuring that, for any $\tau > 0$, the uncertain linear time-delay system in Eqs. (1) and (2) is robustly controllable in sense of Weiss [21].

Theorem. For any $\tau > 0$, the uncertain linear time-delay system in Eqs. (1) and (2) is robustly controllable in sense of Weiss [21], if the matrix $Q$ in Eq. (3) has a full row rank, and if the following condition holds

$$
\sum_{k=1}^{m} \epsilon_k \phi_k < 1,
$$

(8)

where

$$
\phi_k = \begin{cases} 
\mu(-S^{-1}U^H E_k V[I_{n^2}, 0_{n^2 \times n(n-1)}]^T), \text{ for } \epsilon_k \geq 0; \\
-\mu(S^{-1}U^H E_k V[I_{n^2}, 0_{n^2 \times n(n-1)}]^T), \text{ for } \epsilon_k < 0;
\end{cases}
$$

the matrices $E_k$, $S$, $U$ and $V$ are, respectively, defined in Eqs. (6) and (7), and $I_{n^2}$ denotes the $n^2 \times n^2$ identity matrix.
Proof. Since the matrix $Q$ in Eq. (3) has a full row rank and we know that
\[
\text{rank}(Q) = \text{rank}(S^{-1} U^H Q V),
\]
thus, instead of $\text{rank}(\tilde{Q})$, we can discuss the rank of
\[
\left[ I_{n^2} \ 0_{n^2 \times n(m-1)} \right] + \sum_{k=1}^{\bar{m}} \varepsilon_k R_k,
\]
where $R_k = S^{-1} U^H E_k V$, for $k = 1, 2, \ldots, \bar{m}$. Since a matrix has at least rank $n^2$ if it has at least one nonsingular $n^2 \times n^2$ submatrix, a sufficient condition for the matrix in Eq. (10) to have rank $n^2$ is the nonsingularity of
\[
G = I_{n^2} + \sum_{k=1}^{\bar{m}} \varepsilon_k \overline{R}_k,
\]
where $\overline{R}_k = S^{-1} U^H E_k V[I_{n^2}, 0_{n^2 \times n(m-1)}]^T$.

Using the properties in Lemmas 3 and 4, and from Eq. (8), we have
\[
\mu\left(-\sum_{k=1}^{\bar{m}} \varepsilon_k \overline{R}_k\right) = \mu\left(-\sum_{k=1}^{\bar{m}} \varepsilon_k \left(S^{-1} U^H E_k V[I_{n^2}, 0_{n^2 \times n(m-1)}]^T\right)\right)
\leq \sum_{k=1}^{\bar{m}} \mu\left(-\varepsilon_k \left(S^{-1} U^H E_k V[I_{n^2}, 0_{n^2 \times n(m-1)}]^T\right)\right) = \sum_{k=1}^{\bar{m}} \varepsilon_k \phi_k < 1.
\]
Thus, from Lemma 5, we have
\[
\det(G) = \det(I_{n^2} + \sum_{k=1}^{\bar{m}} \varepsilon_k \overline{R}_k) \neq 0.
\]

Hence, the matrix $G$ in Eq. (11) is nonsingular. That is, the matrix $\tilde{Q}$ in Eq. (5) has full row rank $n^2$. Therefore, from the results mentioned-above and Lemma 2, it is ensured that, for any $\tau > 0$, the uncertain linear time-delay system in Eqs. (1) and (2) is robustly controllable in sense of Weiss [21]. Q.E.D.

Remark 1. The proposed sufficient condition in Eq. (8) can give the explicit relationship of the bounds on $\varepsilon_k$ for guaranteeing the robust controllability. In addition, the bounds, that are obtained by using the proposed sufficient condition, on $\varepsilon_k$ are not necessarily symmetric with respect to the origin of the parameter space regarding $\varepsilon_k$, in which $k = 1, 2, \ldots, \bar{m}$. On the other hand, if the parametric uncertainties $\varepsilon_k$ ($k = 1, 2, \ldots, \bar{m}$) are unknown, then the inequality in Eq. (8) can be rewritten as $\varepsilon < (1/\sum_{k=1}^{\bar{m}} \phi_k)$, where $|\varepsilon_k| \leq \varepsilon$, to estimate the maximal bound on parametric uncertainties for guaranteeing the robust controllability.

3. Illustrative examples

In this section, three numerical examples are given to illustrate the applications of the proposed sufficient algebraic criterion. We will also compare the results of the proposed sufficient condition for linear system having no state delay with those obtained from the sufficient criteria of Elizondo and Collado [8] and Cheng and Zhang [4].
\[ \sum_{k=1}^{5} \varepsilon_k \phi_k \leq 0.84065 < 1, \text{ for } \varepsilon_1 \in [-0.1, 0], \varepsilon_2 \in [-0.25, 0], \varepsilon_3 \in [-0.5, 0], \varepsilon_4 \in [-0.15, 0], \text{ and } \varepsilon_5 \in [0, 0.1]. \]  \tag{A.32}

References


