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Oblique circular torus, Villarceau circles, and four types of Bennett linkages

Chung-Ching Lee¹ and Jacques M. Héron²

Abstract
The oblique circular torus (OCT) and its main geometric properties are introduced. Intrinsic vector calculation is utilized to mathematically describe the OCT. The coordinate-free approach leads to the algebraic equation of an OCT in a privileged Cartesian reference frame. The OCT equation is used to confirm a theorem of Euclidean geometry. In a broad category of OCT, through any point five circles can be drawn on the surface, namely the parallel of latitude and four circular generatrices whose planes pass through the OCT center of symmetry. In the special case of a right circular torus, the Villarceau theorem is verified. Next, consider the four RRS open chains whose S spherical-joint centers move on the same OCT and their possible in-parallel assemblies in single-loop RRRS chains. From a category of the foregoing RRRS chains, a new derivation of the amazing Bennett 4R linkage is proposed. Two kinds of Bennett linkages are further verified and each kind contains two enantiomorphic or symmetric linkages. Four types of Bennett linkages associated with one OCT are established by uniquely specifying the link twist as an acute value. Two cases of special type, rectangular and equilateral configurations, are also confirmed.

Keywords
Oblique circular torus, Villarceau circles, circular generatrix, Bennett linkage, enantiomorphic

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Introduction
The surface generated by the rotation of a circle around a fixed axis that belongs to the circle plane is called a torus or toric surface. This is the standard acceptance of the word torus. In fact, the word torus is also used to designate the obtained revolute surfaces when the axis of revolution does not lie on the circle plane. In this way, the shape of the surface may look like a flattened torus. However, the location flattened torus is not mathematically accurate and one may be misled into thinking that it is the toroidal surface obtained by revolving an ellipse that is a flattened circle. A toroidal surface or toroid is a revolute surface swept out by a closed curve, which is not necessarily a circle. When a generatrix is a circle, the surface is called circular toroid.

In what follows, our emphasis is on the oblique circular torus (abbreviated to OCT), which is the revolute surface generated by a circle whose plane intersects the axis of revolution at one point. Moreover, the center of the rotating circle is assumed to belong to the common perpendicular to the rotating circle axis and the axis of revolution. The foot of the perpendicular drawn from the rotating circle center onto the axis of revolution is a fixed point of the OCT axis and the plane that is perpendicular to the axis at this point is the fixed equatorial plane of the OCT. The intersections of the OCT with other fixed planes perpendicular to the OCT axis are called parallels of latitude, which are obviously coaxial circles. The intersections of the OCT with half-planes containing the revolution axis are curves called meridians. Clearly, the meridians are congruent by rotation around the revolution axis. In the standard torus, the meridians are circles, but in an OCT, they are neither circles nor ellipses. The standard torus is a special case of OCT and can be termed a right circular torus. In 1848, Yvon Villarceau discovered an amazing
geometric property of the torus: a plane that is bitangent to a torus intersects the surface along two congruent circles. The circles are symmetric with respect to a plane containing the axis of revolution. These two circles are called the Villarceau circles. Our paper verifies that Villarceau’s theorem corresponds to a special case of a general theorem on the OCT. Through any point of a broad category of OCT, one can draw on the OCT surface five circles, which are the parallel of latitude, and two couples of congruent circular generatrices whose planes pass through the OCT center of symmetry. A plane that is bitangent to an OCT intersects the surface along a couple of congruent circles, which generalize the Villarceau circles. The intersection of an OCT by a plane belonging to another family of planes, which is not bitangent to the OCT, is also a couple of congruent circles. With the help of this theorem of Euclidean geometry on OCT, this article presents a new way to derive the well-known Bennett linkage, which is movable with one-degree-of-freedom (1-DoF) of finite mobility.

The Bennett linkage is the only movable four-revolute chain with non-parallel and non-intersecting revolute R joint axes. The link twist angles and the link lengths, which are mathematically related by one constraint equality, characterize it. It can be viewed as a warped hinged parallelogram and its opposite link lengths and link twist angles are equal. G. T. Bennett first discovered this four-revolute mechanism in 1903. E. Borel found it independently only one year later after Bennett. R. Bricard also noticed this outstanding finding. Specifically, this linkage disobeys the Chebyshev–Gruebler–Kuzbach formula and it was qualified as paradoxical in the terminology of J. M. Hervé scientific work. Paradoxical mobility is subject to geometric conditions that cannot be expressed without using Euclidean (or Pythagorean) metric. The geometric figure of four points and four segments of straight lines, whose alternate sides are equal in length, was called isogram by Bennett. The location of skew isogram mechanism was used to describe this linkage. The connection between the Bennett linkage and the Villarceau theorem was first detected by Bricard and was confirmed by F. E. Myard. A noteworthy work of Krames provided further insights into the relation between Bennett’s linkage and Villarceau’s circles. At the same time, the motion generated by the coupler of Bennett mechanism is line symmetric and the line of symmetry lies on a regulus of a hyperboloid of one sheet. However, the signs of Bennett equality are not discussed in Krames’ paper. Besides, the Cartesian equation of an OCT appears nowhere and the Bennett relation is not derived by logical deduction from the OCT equation. The existence of two categories of OCT is also ignored. Two kinds of Bennett mechanisms were elucidated in a published work. Hereinafter, a new geometric derivation of Bennett 4R linkage will be done by considering RRS open chains whose spherical S joint center moves on the same OCT.

This article proceeds as follows. First, an OCT is the surface trajectory of a point in a 2-dof motion generated by a serial concatenation of two revolute R joints. From a parametric representation obtained through the product of two rotations, a Cartesian equation of an OCT is derived. That equation further provides three other ways to generate the same surface by rotating circles. The four circular generatrices of an OCT are associated with four RRS open chains producing the same Cartesian equation of an OCT. Some assemblies of two RRS open chains yield a geometrical derivation of four types of Bennett linkages having the same link lengths and the same absolute value of the link twist angles. An appropriate convention for orientating vectors and angles is adopted to discuss special cases comprehensively as well.

The OCT

Definition of an OCT

Let us consider the serial concatenation of two revolute R pairs shown in Figure 1, which is sometimes called also a two-hinge dyad in the literature on mechanism theory. The open RR chain includes three rigid bodies connected in series by two revolute R joints or hinges. Each of the two R joints (or R pairs) is characterized by its axis and each axis is determined by the datum of any one of its points and a unit vector parallel to the axis in an initial home configuration of the RR chain. Assuming that the two axes are not parallel, there is a unique common perpendicular to the two axes. It is convenient to choose two points on the common perpendicular to specify the two R axes. When the first fixed R pair is locked, a point P belonging to the end body of a two-hinge dyad traces a circle on a plane, which is perpendicular to the second R axis and is located at a distance d of the common perpendicular. The distance d is called offset. When the fixed R is unlocked, the point P traces a surface of revolution, which is called

Figure 1. The circular toroid obtained as a surface trajectory in a RR dyad.
a general circular toroid, as shown in Figure 1. In the noteworthy paper, Fichter and Hunt proved that a plane that is bitangent to a general circular toroid intersects the surface along a couple of congruent circles. The pertinent proof is established by using the complex projective completion of the Euclidean space. In what follows, one will not consider imaginary geometric entities but will focus on the special case with a zero offset.

When the offset (d) is zero and two R axes are not twisted by a right angle (|β| ≠ 90°), the circular toroid becomes an OCT depicted in Figure 2. The case (|β| = 90°) corresponds to the right circular torus that is the standard torus. An OCT is produced by revolving an inclined circle (circular generatrix) whose plane contains the common perpendicular of the circle axis and the revolution axis. It is proved that, generally, the revolution of other circular generatrices lying on planes containing the OCT center produces the same OCT. Because of the obvious symmetry of the revolute surface with respect to any plane containing the revolution axis, a second generatrix is simply the plane-symmetric of the first one. In the case (|β| = 90°), the foregoing two circular generatrices are meridians and the other two circles are those of Villarceau’s theorem.

Refer to Figure 2; (O, i, j, k) is a Cartesian ortho-normal frame of reference whose origin O is the intersection of the first R axis and the common perpendicular of the two R axes. The unit vector k is parallel to the first R axis, which is characterized by (O, k). The unit vector i is parallel to the common perpendicular of the two R axes. The point B is the intersection of the common perpendicular and the second R axis. The vector (OB) = B − O is equal to bi. The number b, b > 0, is the length (or Euclidean norm) ||OB|| of the bar (OB). The unit vector s is parallel to the axis of the second R pair. In the concatenation of two R pairs, the axes are twisted by an angle β around the common normal OB. It is worth noticing that through the usual convention of handedness, the vector i orients the angle β. An observer

(Ampère’s manikin in electromagnetism theory) lying along the axis (O, i) with O at his feet and i going from his feet towards his head sees the counterclockwise rotations with positive angles whereas the clockwise rotations have negative angles. It is important to note that the same given rotation has two representations. If one is a rotation of angle β around an axis parallel to i, then the other one is the rotation of angle −β around (−i). That way, without loss of generality, the angle β of twist in the oriented bar (OB) can be, and therefore is assumed to be acute, 0 < |β| < π/2. Specifying the link twist as an acute angle will uniquely determine the proper twist of link.

In this RR chain, the first R pair has the fixed axis (O, k) and the second R pair has the axis (B, s). A point M is attached at the end body in such a way that the segment (BM) is collinear with (OB) at the initial home posture of the RR chain: vector(BM) = M − B = ai with a = ||BM||. One can notice that [β] is the angle between the equatorial plane of the OCT and the planes of the circular trajectories of M when the fixed R is locked. Assuming that the fixed axis (O, k) is vertical, the angle [β] = [dk, s] is the common slope of a family of planes, which contain a circular generatrix of the OCT.

Cartesian equation of an OCT

Following two steps, one can obtain the variable position of the point M, which moves with respect to the fixed frame (O, i, j, k), and its 2-dof motion mathematically. In the first step, the second R rotates around the axis (B, s) with an angle ϕ while the first R keeps its home pose (θ = 0); the initial position M₀ of M becomes Mᵥ, which is expressed by

\[
M₀ → Mᵥ = B + \exp(ϕ s×)(BM₀)
\]  

(1)

Using

\[
s = \exp(β i×)k = −Sin β j + Cos β k
\]  

(2a)

\[
v = \exp(β i×)j = Cos β j + Sin β k = s × i
\]  

(2b)

\[
(\mathbf{OB}) = bi, \quad (\mathbf{BM₀}) = (a + b)i
\]  

(2c)

then,

\[
(\mathbf{OMᵥ}) = (\mathbf{OB}) + (\mathbf{BMᵥ}) = (\mathbf{OB}) + \exp(ϕ s×)(\mathbf{BM₀})
\]

\[
= bi + a(Sin ϕ s × i + Cos ϕ i)
\]

\[
= bi + a(Sin ϕ v + Cos ϕ i)
\]

\[
= bi + a(Sin ϕ (Cos β j + Sin β k) + Cos ϕ i)
\]

\[
= (b + aCos ϕ)i + aSin ϕ Cos β j + aSin ϕ Sin β k
\]  

(3)

This is a parametric representation of the circular trajectory of Mᵥ with only the rotation in the second R pair.

Figure 2. An oblique circular torus obtained as a surface trajectory in a RR dyad.
In the second step, once the second R pair is locked (angle $\varphi$ keeps its value), the first R pair rotates around the axis $(O, k)$ by an angle $\theta$ and the point position $M_y$ is transformed into $M_T$.

$$M_y \rightarrow M_T = O + \exp(\theta k \times)(OM_y)$$  \hspace{1cm} (4)

Note that the rotation around $(O, k)$ with an angle $\theta$ yields

$$i \rightarrow \cos \theta i + \sin \theta j$$  \hspace{1cm} (5a)

$$j \rightarrow -\sin \theta i + \cos \theta j$$  \hspace{1cm} (5b)

$$k \rightarrow k$$  \hspace{1cm} (5c)

Hence,

$$(OM_T) = (b + a \cos \varphi)(\cos \theta i + \sin \theta j)$$

$$+ a \sin \varphi \cos \beta (-\sin \theta i + \cos \theta j)$$

$$+ a \sin \varphi \sin \beta k$$  \hspace{1cm} (6)

The coordinates $(X, Y, Z)$ of the point $M_T$ belonging to the OCT are functions of the two parameters $\theta$ and $\varphi$, which are angles of rotation in the open RR chain:

$$X = (b + a \cos \varphi)\cos \theta - a \sin \varphi \cos \beta \sin \theta$$

$$Y = (b + a \cos \varphi)\sin \theta + a \sin \varphi \cos \beta \cos \theta$$

$$Z = a \sin \varphi \sin \beta$$  \hspace{1cm} (7)

Eliminating $\theta$ and $\varphi$ from equation (7) yields the Cartesian equation of the surface as follows,

$$(X^2 + Y^2 + Z^2 - b^2 - a^2)^2 + \frac{4b^2}{\sin^2 \beta} Z^2 = 4a^2 b^2$$  \hspace{1cm} (8)

This is an algebraic equation for the surface trajectory of $M$ located at the end of the last link.

**Other circular generatrices of an OCT**

The surface modeled by equation (8) is an OCT. From this quartic equation, one can notice that replacing $b$ by $a$ and $\beta$ by a second angle $\alpha$ does not change the surface equation provided that the equality $a^2 \sin^2 \beta = b^2 \sin^2 \alpha$ is satisfied. Consequently, another RR bar with a common perpendicular of length $a$ (instead of $b$) and a twist angle $\alpha$ verifying $\sin^2 \alpha = (a/b)^2 \sin^2 \beta$ produces the same surface trajectory for a point located at the distance $b$ of the second R axis (instead of $a$ in the original RR system). For a given $(a/b)^2 \sin^2 \beta$, when assuming $(a/b)^2 \sin^2 \beta \leq 1 \Leftrightarrow \alpha \equiv (a/b) \sin \beta \leq 1 \Leftrightarrow \sin \beta \leq b/a$, the equation of unknown $\alpha$, $\sin^2 \alpha = (a/b)^2 \sin^2 \beta$, has two solutions that are of opposite signs under the assumption $\alpha \in (-\pi/2, +\pi/2)$. Noteworthy is the fact that all the systems of arbitrary data $(a, b, \beta)$ are not valid for the replacement of $b$ by $a$ and $\beta$ by $\alpha$. For instance, if $b = 1$ and $a = 2$, then the inequality $\sin \beta \leq 1/2 \Leftrightarrow |\beta| \leq \pi/3$ limiting the possible choices of $|\beta|$ and $|\beta| \geq \pi/3$ gives OCTs, which do not allow the foregoing replacement. When $b/a \geq 1 \Leftrightarrow b \geq a$, the surface is a ring OCT or a horn OCT; the inequality $\sin (\beta) \leq b/a$ is always verified and $|\alpha|$ is equal to $\sin^{-1} (a/b) \sin \beta$. Here, $\sin^{-1}$ denotes the inverse function of the sine function rather than $1/\sin$. This inverse function is multivalued and becomes single valued when its value is restricted to belonging to the domain $(-\pi/2, \pi/2)$. Hence, under the assumption $\alpha \in (-\pi/2, \pi/2)$, $|\alpha| = \sin^{-1} [(a/b) \sin \beta]$ yields one value for $|\alpha|$. If $b/a < 1 \Leftrightarrow b < a$, then the surface belongs to a category of OCT divided into two subcategories. Either the angle $|\beta|$ is small enough to satisfy $\sin (\beta) \leq b/a$ or $|\beta|$ is such as $\sin (\beta) > b/a$; there is no angle $|\alpha|$ satisfying $a^2 \sin^2 \beta = b^2 \sin^2 \alpha$. Consequently, when the given system $(a, b, \beta)$ of two lengths and one angle is appropriate, there are four RR open chains producing the same OCT for a point $M$ of the outermost body in the triplet of rigid bodies. The metric properties of the four chains are symbolized by $(b, \beta; a)$, $(a, \alpha; b)$, $(b, -\beta; a)$, and $(a, -\alpha; b)$ where the combination $(b, \beta; a)$ characterizes the original mechanism used to obtain the OCT equation.

**Theorems on OCT circular generatrices**

In Figure 3, through an arbitrary point $Q$ on the surface of the OCT, five circles drawn to the OCT surface intersect. They are the parallel of latitude and four circular generatrices. One can state a theorem of Euclidean geometry: for a broad family of OCTs, an OCT has four circular generatrices, which lie on planes containing the OCT center. From the datum of one circular generatrix of an OCT, the other three circular generatrices can be derived.

That theorem is complemented by the property: any plane of slope, either $|\alpha|$ or $|\beta|$ and passing through the OCT center that is the origin $O$, intersects the surface along a couple of congruent circles. As the

**Figure 3.** Five circles on the surface of OCT.
Conclusions

The OCT is a revolute surface, which is less special than the standard torus but is more special than the general circular toroid. From the OCT equation, two categories of OCTs are detected. In a broad category of OCTs, the revolute surface has four circular generatrices whose planes contain the OCT center of symmetry. Because of the symmetry about any plane containing the axis of revolution, only two generatrices are not congruent circles amid the four circles. The Villarceau theorem corresponds to a special case of OCT. The mechanical generation of the OCTs belonging to the appropriate category leads to special RRRS chains in a first step and to the Bennett 4R linkage in a second step. From the OCT equation, a new formulation of the so-called Bennett equality (or index) tying link lengths and twist angles in the four bars of Bennett linkage is derived and proposed to be an improvement of what is available in already published literature. Based on the specification of a coherent convention for orienting the bars and the twist angles together with the choice of an appropriate range for the twist angles, the discussion of the improved Bennett equality results in the discrimination of four related types for the non-equilateral Bennett linkages. The four types are sorted into two kinds; each kind contains a pair of enantiomorphic linkages. Apparently, for the first time, the enantio-merism in mechanisms is taken into account, explicitly. There are only two types of rectangular Bennett linkages and one type of non-degenerated equilateral Bennett linkages. Probably, the presentation of hybrids of Bennett linkages needs to be revisited.

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